**A**

**2012 MS**

**Test Paper Code: MS**

**Time: 3 Hours** **Maximum Marks: 300**

**INSTRUCTIONS**

1. This question-cum-answer booklet has **32** pages and has **25** questions. Please ensure that the copy of the question-cum-answer booklet you have received contains all the questions.
2. Write your **Registration Number, Name and the name of the Test Centre** in the appropriate space provided on the right side.
3. Write the answers to the objective questions against each Question No. in the **Answer Table for Objective Questions**, provided on Page No. **7**. Do not write anything else on this page.
4. Each objective question has **4 choices** for its answer: (A), (B), (C) and (D). Only **ONE** of them is the correct answer. There will be **negative marking** for wrong answers to objective questions. The following marking scheme for objective questions shall be used:
   (a) For each correct answer, you will be awarded **6 (Six)** marks.
   (b) For each wrong answer, you will be awarded **- 2 (Negative two)** marks.
   (c) Multiple answers to a question will be treated as a wrong answer.
   (d) For each un-attempted question, you will be awarded **0 (Zero)** mark.
   (e) Negative marks for objective part will be carried over to total marks.
5. Answer the subjective question only in the space provided after each question.
6. Do not write more than one answer for the same question. In case you attempt a subjective question more than once, please cancel the answer(s) you consider wrong. Otherwise, the answer appearing last only will be evaluated.
7. All answers must be written in blue/black/blue-black ink only. Sketch pen, pencil or ink of any other colour should not be used.
8. All rough work should be done in the space provided and scored out finally.
9. No supplementary sheets will be provided to the candidates.
10. **Clip board, log tables, slide rule, calculator, cellular phone and electronic gadgets in any form are NOT allowed.**
11. The question-cum-answer booklet must be returned in its entirety to the Invigilator before leaving the examination hall. Do not remove any page from this booklet.
12. Refer to special instructions/useful data on the reverse.

**2012 MS**

**READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY**

| REGISTRATION NUMBER | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| Name: | | | | | | |
| Test Centre: | | | | | | |

**Do not write your Registration Number or Name anywhere else in this question-cum-answer booklet.**

I have read all the instructions and shall abide by them.

……………………………………………..

Signature of the Candidate

I have verified the information filled by the Candidate above.

……………………………………………..

Signature of the Invigilator

**Special Instructions/ Useful Data**

1. $\mathbb{R}$ : Set of all real numbers.
2. i.i.d. : independent and identically distributed.
3. $N(\mu, \sigma^2)$ : Normal distribution with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$.
4. For a fixed $\lambda > 0$, $X \sim Exp(\lambda)$ means that the probability density function of random variable $X$ is
   $$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & if \;\; x > 0, \\ 0 & otherwise. \end{cases}$$
5. $U(a,b)$ : Uniform distribution on $(a, b)$, $-\infty < a < b < \infty$.
6. $B(n, p)$ : Binomial distribution with parameters $n \in \{1, 2, ...\}$ and $p \in (0,1)$.
7. $E(X)$ : Expectation of $X$.
8. $F_{m,n}$ : $F$ distribution with $m$ and $n$ degrees of freedom.
9. $\overline{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ is the sample mean based on $(x_1, ..., x_n)$.
10. $\alpha$ = P [type I error] and $\beta$ = P [type II error]
11. $H_0$ : Null Hypothesis, $H_1$ : Alternative Hypothesis

**Useful data**

$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ , where $\Phi(z)$ is cumulative distribution function of $N(0, 1)$.

$\Phi(1.28) = 0.900$, $\Phi(1.65) = 0.950$, $\Phi(1.96) = 0.975$,
$\Phi(2.33) = 0.990$, $\Phi(2.58) = 0.995$.

**IMPORTANT NOTE FOR CANDIDATES**

- **Questions 1-15 (objective questions) carry *six* marks each and questions 16-25 (subjective questions) carry *twenty one* marks each.**
- **Write the answers to the objective questions in the *Answer Table for Objective Questions* provided on page 7 only.**

Q.1 An eigenvector of the matrix $M = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ is

(A) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (B) $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (C) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (D) $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$

Q.2 The volume of the solid of revolution generated by revolving the area bounded by the curve $y = \sqrt{x}$ and the straight lines $x = 4$ and $y = 0$ about the $x-$axis, is

(A) $2\pi$ (B) $4\pi$ (C) $8\pi$ (D) $12\pi$

Q.3 Let $I = \int_0^1 \int_{x^2}^{2-x} xy \; dy\, dx.$ The change of order of integration in the integral gives $I$ as

(A) $I = \int_0^1 \int_0^{\sqrt{y}} xy \; dx\, dy + \int_1^2 \int_0^{2-y} xy \; dx\, dy.$

(B) $I = \int_0^1 \int_0^{2-y} xy \; dx\, dy + \int_1^2 \int_0^{2-y} xy \; dx\, dy.$

(C) $I = \int_0^1 \int_0^{\sqrt{y}} xy \; dx\, dy + \int_0^1 \int_0^{2-y} xy \; dx\, dy.$

(D) $I = \int_0^1 \int_0^{2-y} xy \; dx\, dy + \int_1^2 \int_0^{\sqrt{y}} xy \; dx\, dy.$

Q.4 Let $L = \lim_{n\to\infty} n \left[ f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{k}{n}\right) - k\, f(0) \right]$, where $k$ is a positive integer. If $f(x) = \sin x$, then $L$ is equal to

(A) $\frac{(k+1)(k+2)}{6}$ (B) $\frac{(k+1)(k+2)}{2}$ (C) $\frac{k(k+1)}{2}$ (D) $k(k+1)$

Q.5 Let $f(x,y) = \begin{cases} \sqrt{x^2+y^2}\,\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Then at the point $(0,0)$,

(A) $f$ is continuous and $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist.

(B) $f$ is continuous and $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ do not exist.

(C) $f$ is not continuous and $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist.

(D) $f$ is not continuous and $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ do not exist.

Q.6 Let $\{a_n\}$ be a real sequence converging to $a$, where $a > 0$. Then

(A) $\sum_1^\infty a_n$ converges but $\sum_1^\infty \dfrac{a_n}{n}$ diverges.

(B) $\sum_1^\infty a_n$ diverges but $\sum_1^\infty \dfrac{a_n}{n}$ converges.

(C) Both $\sum_1^\infty a_n$ and $\sum_1^\infty \dfrac{a_n}{n}$ converge.

(D) Both $\sum_1^\infty a_n$ and $\sum_1^\infty \dfrac{a_n}{n}$ diverge.

Q.7 A four digit number is chosen at random. The probability that there are exactly two zeros in that number is

(A) 0.73 (B) 0.973 (C) 0.027 (D) 0.27

Q.8 A person makes repeated attempts to destroy a target. Attempts are made independent of each other. The probability of destroying the target in any attempt is 0.8. Given that he fails to destroy the target in the first five attempts, the probability that the target is destroyed in the $8^{th}$ attempt is

(A) 0.128 (B) 0.032 (C) 0.160 (D) 0.064

Q.9 Let the random variable $X \sim B(5, p)$ such that $P(X=2) = 2P(X=3)$. Then the variance of $X$ is

(A) $\dfrac{10}{3}$ (B) $\dfrac{10}{9}$ (C) $\dfrac{5}{3}$ (D) $\dfrac{5}{9}$

Q.10 Let $X_1,\ldots,X_8$ be i.i.d. $N(0,\sigma^2)$ random variables. Further, let $U = X_1 + X_2$ and $V = \sum_{i=1}^{8} X_i$. The correlation coefficient between $U$ and $V$ is

(A) $\frac{1}{8}$ (B) $\frac{1}{4}$ (C) $\frac{3}{4}$ (D) $\frac{1}{2}$

Q.11 Let $X \sim F_{8,15}$ and $Y \sim F_{15,8}$. If $P(X > 4) = 0.01$ and $P(Y \le k) = 0.01$, then the value of $k$ is

(A) 0.025 (B) 0.25 (C) 2 (D) 4

Q.12 Let $X_1,\ldots,X_n$ be i.i.d. $Exp(1)$ random variables and $S_n = \sum_{i=1}^{n} X_i$. Using the central limit theorem, the value of $\lim_{n\to\infty} P(S_n > n)$ is

(A) 0 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 1

Q.13 Let the random variable $X \sim U(5, 5+\theta)$. Based on a random sample of size 1, say $X_1$, the unbiased estimator of $\theta^2$ is

(A) $3(X_1-5)^2$ (B) $\frac{X_1^2-5}{12}$ (C) $3(X_1+5)^2$ (D) $\frac{X_1^2+5}{12}$

Q.14 Let $X_1,\ldots,X_n$ be a random sample of size $n$ from $N(\mu,16)$ population. If a 95% confidence interval for $\mu$ is $\left[\bar{X} - 0.98,\ \bar{X} + 0.98\right]$, then the value of $n$ is

(A) 4 (B) 16 (C) 32 (D) 64

Q.15 A coin is tossed 4 times and $p$ is the probability of getting head in a single trial. Let $S$ be the number of head(s) obtained. It is decided to test
$H_0 : p = \frac{1}{2}$ against $H_1 : p \ne \frac{1}{2}$, using the decision rule : Reject $H_0$ if $S$ is 0 or 4. The probabilities of Type I error ($\alpha$), and Type II error ($\beta$) when $p = \frac{3}{4}$, are

(A) $\alpha = \frac{1}{4},\quad \beta = \frac{87}{128}$

(B) $\alpha = \frac{1}{8},\quad \beta = \frac{87}{128}$

(C) $\alpha = \frac{1}{8},\quad \beta = \frac{41}{256}$

(D) $\alpha = \frac{1}{4},\quad \beta = \frac{41}{256}$

**Space for rough work**

Space for rough work

**Space for rough work**

## *Answer Table for Objective Questions*

**Write the Code of your chosen answer only in the 'Answer' column against each** Question Number**. Do not write anything else on this page.**

| **Question Number** | **Answer** | **Do not write in this column** |
|---|---|---|
| 01 | | |
| 02 | | |
| 03 | | |
| 04 | | |
| 05 | | |
| 06 | | |
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| 15 | | |

**FOR EVALUATION ONLY**

| | | | |
|---|---|---|---|
| Number of Correct Answers | | Marks | ( + ) |
| Number of Incorrect Answers | | Marks | ( – ) |
| Total Marks in Questions 1-15 | | | (    ) |

Q.16 (a) Find the value(s) of $\lambda$ for which the following system of linear equations

$$\begin{pmatrix} \lambda & 1 & 1 \\ 1 & \lambda & 1 \\ 1 & 1 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(i) has a unique solution,
(ii) has infinitely many solutions,
(iii) has no solution. (9)

(b) Let $a_1 = 2, b_1 = 1$ and for $n \geq 1$, $a_{n+1} = \dfrac{a_n + b_n}{2}$, $b_{n+1} = \dfrac{2a_n b_n}{a_n + b_n}$. Show that

(i) $b_n \leq a_n$ for all $n$,
(ii) $b_{n+1} \geq b_n$ for all $n$,
(iii) the sequences $\{a_n\}$ and $\{b_n\}$ converge to the same limit $\sqrt{2}$. (12)

Q.17 (a) Solve: $\left(x^2y^3 + xy\right)dy = dx$. (9)

(b) Find the general solution of the differential equation $\left(D^2 - 4D + 4\right)y = x\sin 2x$, where $D \equiv \dfrac{d}{dx}$. (12)

Q.18 (a) Find all the critical points of the function $f(x, y) = x^3 + y^3 + 3xy$ and examine those points for local maxima and local minima. (9)

(b) If $f$ is a continuous real-valued function on $[0,1]$, show that there exists a point $c \in (0,1)$ such that $\int_0^1 x\, f(x)\, dx = \int_c^1 f(x)\, dx$. (12)

Q.19

(a) Evaluate the triple integral: $\int_{z=0}^{z=4} \int_{x=0}^{x=2\sqrt{z}} \int_{y=0}^{y=\sqrt{4z-x^2}} dy\,dx\,dz$ . (9)

(b) Let $M = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. If $M^{-1} = \frac{5}{4}I + kM + \frac{1}{4}M^2$, where $I$ is the identity matrix of order 3, find the value of $k$. Hence or otherwise, solve the system of equations: $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. (12)

Q.20 (a) Let $N$ be a random variable representing the number of fair dice thrown with probability mass function $P(N=i)=\frac{1}{2^i}$, $i=1,2,\ldots$. Let $S$ be the sum of the numbers appearing on the faces of the dice. Given that $S=3$, what is the probability that 2 dice were thrown ? (9)

(b) Let $X \sim N(0,1)$ and $Y = X + |X|$. Find $E(Y^3)$. (12)

Q.21 Let $Y \sim N(\mu_y, \sigma_y^2)$ and $Y = \ln X$ .

(a) Find the probability density function of the random variable $X$ and the median of $X$. (9)

(b) Find the maximum likelihood estimator of the median of the random variable $X$ based on a random sample of size $n$. (12)

Q.22 (a) A random variable $X$ has probability density function

$$f(x) = \alpha\, x e^{-\beta^2 x^2}, \quad x > 0, \; \alpha > 0, \; \beta > 0.$$

If $E(X) = \dfrac{\sqrt{\pi}}{2}$, determine $\alpha$ and $\beta$. (9)

(b) Let $X$ and $Y$ be two random variables with joint probability density function

$$f(x, y) = \begin{cases} e^{-y} & \text{if } 0 \le x \le y < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the marginal density functions of $X$ and $Y$.

(ii) Examine whether $X$ and $Y$ are independent.

(iii) Find $Cov(X, Y)$. (12)

Q.23 (a) Let $X_1, \ldots , X_n$ be a random sample from $Exp\left(\frac{1}{\theta}\right)$ population. Obtain the Cramer – Rao lower bound for the variance of an unbiased estimator of $\theta^2$. (9)

(b) Let $X_1, \ldots, X_n$ $(n > 4)$ be a random sample from a population with mean $\mu$ and variance $\sigma^2$. Consider the following estimators of $\mu$

$$U = \frac{1}{n}\sum_{i=1}^{n} X_i , \qquad V = \frac{1}{8}X_1 + \frac{3}{4(n-2)}(X_2 + \cdots + X_{n-1}) + \frac{1}{8}X_n .$$

(i) Examine whether the estimators $U$ and $V$ are unbiased.

(ii) Examine whether the estimators $U$ and $V$ are consistent.

(iii) Which of these two estimators is more efficient? Justify your answer. (12)

Q.24 Let $X_1, \ldots, X_n$ be a random sample from a Bernoulli population with parameter $p$.

(a) (i) Find a sufficient statistic for $p$.

(ii) Consider an estimator $U(X_1, X_2)$ of $\frac{p(1-p)}{n}$ given by

$$U(X_1, X_2) = \begin{cases} \dfrac{1}{2n} & \text{if } \; X_1 + X_2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Examine whether $U(X_1, X_2)$ is an unbiased estimator.

(9)

(b) Using the results obtained in (a) above and Rao – Blackwell theorem, find the uniformly minimum variance unbiased estimator (UMVUE) of $\frac{p(1-p)}{n}$. (12)

Q.25 (a) Let $X_1, \ldots, X_n$ be a random sample from the population having probability density function

$$f(x,\theta) = \begin{cases} \dfrac{2x}{\theta^2} e^{-\frac{x^2}{\theta^2}} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Obtain the most powerful test for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ $(\theta_1 < \theta_0)$. (9)

(b) Let $X_1, \ldots, X_n$ be a random sample of size *n* from $N(\mu, 1)$ population. To test $H_0 : \mu = 5$ against $H_1 : \mu = 4$, the decision rule is : Reject $H_0$ if $\overline{x} \le c$. If $\alpha = 0.05$ and $\beta = 0.10$, determine *n* (rounded off to an integer) and hence $c$. (12)

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| 2012 MS<br>Objective Part<br>( Question Number 1 – 15) | |
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| **Total Marks** | **Signature** |
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| Subjective Part | | | | | |
|---|---|---|---|---|---|
| **Question Number** | **Marks** | | **Question Number** | **Marks** | |
| **16** | | | **21** | | |
| **17** | | | **22** | | |
| **18** | | | **23** | | |
| **19** | | | **24** | | |
| **20** | | | **25** | | |
| **Total Marks in Subjective Part** | | | | | |

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| **Total (Subjective Part) :** | |
| **Grand Total :** | |
| **Total Marks (in words) :** | |
| **Signature of Examiner(s) :** | |
| **Signature of Head Examiner(s) :** | |
| **Signature of Scrutinizer :** | |
| **Signature of Chief Scrutinizer :** | |
| **Signature of Coordinating Head Examiner :** | |